Sheaves on Proj

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12.11.2025 Lecture continued

Let $R = \bigoplus_{j \in \mathbb{N}} R_j$ be an $\mathbb{N}$-graded ring. We constructed a scheme $(X, \mathcal{O}_X) = (\mathrm{Proj}(R), \mathcal{O}_{\mathrm{Proj}(R)})$; Set $R_+ = \bigoplus_{j>0} R_j$

Recall $\mathrm{Proj}(R) = \{ p \mid p \text{ homogeneous prime}, p \not\supseteq R_+ \}$

The top on $\mathrm{Proj}(R)$ has open basis $\{D_+(f)\}$ f homo of +ve deg

$D_+(f) = \{ p \in \mathrm{Proj}(R) \mid f \notin p \}$

$\mathcal{O}_X(D_+(f)) \cong (R[1/f])_0$.

§ Recall, a $\mathbb{Z}$-graded R mod is $M \in \mathrm{Mod}_R$ such that $M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda$ as an ab group

and $R_{\lambda_1} \cdot M_{\lambda_2} \subseteq M_{\lambda_1 + \lambda_2}$, $\forall \lambda_1 \in \mathbb{N}, \lambda_2 \in \mathbb{Z}$.

- A graded R-lin map (or simply a graded map) between two graded mods is an R lin map $\varphi: M \to N$ s.t $\varphi(M_\lambda) \subseteq N_\lambda$ $\forall \lambda \in \mathbb{Z}$.
- The cat of $\mathbb{Z}$-graded R-mods with graded R-lin map forms a cat denoted Mod_R^{gr}

Given $M \in \mathrm{Mod}_R^{gr}$, we construct a sheaf of $\mathcal{O}_X$-mods, denoted $\tilde{M}$ on $X = \mathrm{Proj}\, R$

Caution: This $\tilde{M}$ is not the q coh sheaf $\tilde{M}$ on $\mathrm{Spec}(R)$

- Note given a mult closed set S of homogeneous elts of R, $S^{-1}R$ is a $\mathbb{Z}$-graded ring. $S^{-1}M$ is a $\mathbb{Z}$-graded mod / $S^{-1}R$.

$$(S^{-1}R)_\lambda = \{ r/s \in S^{-1}R \mid r, s \text{ homo}, \deg r - \deg s = \lambda \}$$

$$(S^{-1}M)_\lambda = \{ m/s \in S^{-1}M \mid m, s \text{ homo}, \deg m - \deg s = \lambda \}$$

- For $p \in \mathrm{Proj}(R)$, set $S_p^* =$ homo elts of $R \setminus p$

$M_{(p)} = (S_p^{*-1} M)_0$

Def of $\tilde{M}$: Consider the fou sheaf of $\mathcal{O}_X$-mod on X;

$\tilde{M}(U) = \{$ set maps $s: U \to \coprod_{p \in U} M_{(p)} \mid$
- $s(p) \in M_{(p)}$
- for every $p \in U$, $\exists$ a homogeneous f of +ve deg, s.t $f \notin p$ and $m \in M_{\deg f}$ s.t $\forall q \in D_+(f) \cap U$ $s(q) = m/f \in M_{(q)}$ $\}$

End of 12.11.2025 lecture

Recall: $\tilde{R} = \mathcal{O}_X$, So for $s \in \mathcal{O}_X(U)$, $t \in \tilde{M}(U)$

$(s\,t)(q) = \underset{R_{(q)}}{s(q)} \cdot \underset{M_{(q)}}{t(q)} \in M_{(q)}$ $\forall q \in U$

Thm: 1) $\tilde{M}$ is a sheaf for any $M \in \mathrm{Mod}_R^{gr}$.
For $p \in \mathrm{Proj}\,R$, the natural map $M_{(p)} \to (\tilde{M})_p$ is an isom, with inverse given by $s \mapsto s(p)$ for a section $s \in \tilde{M}(U)$, $p \in U$.

2) Given a graded R-lin map $\varphi: M \to N$, naturally have $\tilde{\varphi}: \tilde{M} \to \tilde{N}$

for $t \in \tilde{M}(U)$, $p \in U$: $\tilde{\varphi}_U(t)(p) = \varphi_{(p)}(t(p))$; $\varphi_{(p)}: M_{(p)} \to N_{(p)}$

3) $\widetilde{\varphi_1 \cdot \varphi_2} = \tilde{\varphi}_1 \cdot \tilde{\varphi}_2$, $\widetilde{id_M} = id_{\tilde{M}}$

So $M \mapsto \tilde{M}$ is gives a functor $\mathrm{Mod}_R^{gr} \to \mathrm{Mod}_{\mathcal{O}_X}$.

Thm: For $M \in \mathrm{Mod}_R^{gr}$; $\tilde{M} \in \mathrm{QCoh}(X)$.
Moreover $\Gamma(D_+(f), \tilde{M}) \cong (M[1/f])_0$ for every homo elt f of +ve deg.

every homo ...

Pf. Since being qcoh is a local property, it's enough to check that for any homo elt f of +ve deg $\tilde{M}|_{D_+(f)}$ is qcoh.

Set $\Gamma = M[1/f]_0$. The identity map $\Gamma \to \Gamma(D_+(f), \tilde{M})$ gives an $\mathcal{O}_{D_+(f)}$-line map

$\ast \quad \tilde{\Gamma} \longrightarrow \tilde{M}|_{D_+(f)}$ as $\tilde{\Gamma}$ is qcoh on the affine $D_+(f)$.

Claim. $\ast$ is an isom.

Pf. We check isom at the stalks at $\mathfrak{p} \in D_+(f)$.

Recall that $D_+(f) \xrightarrow{\sim} \operatorname{Spec}(R[1/f]_0)$

$\mathfrak{q} \longmapsto \mathfrak{q}R[1/f] \cap R[1/f]_0 =: \mathfrak{q}_a$

The map induced by $\ast : (M[1/f]_0)_{\mathfrak{p}_a} \simeq \tilde{\Gamma}_{\mathfrak{p}_a} \longrightarrow \tilde{M}_{\mathfrak{p}} \xrightarrow{\sim} M_{(\mathfrak{p})}$ $\ast\ast$

$$g \in S_{\mathfrak{p}}^{\ast}, \quad \frac{m/f^{e_1}}{g/f^{e_2}} \longmapsto \frac{m f^{e_2}}{g f^{e_1}}$$

Note $(M[1/f]_0)_{\mathfrak{p}_a} = [(S_{\mathfrak{p}}^{\ast})^{-1} M[1/f]]_0 = (S_{\mathfrak{p}}^{\ast -1} M)_0$

So $\ast\ast$ is an isom

Def. $F: \mathcal{A} \to \mathcal{B}$ be a functor between two abelian categories

① F is called <u>additive</u> if for any $a, b \in \mathcal{A}$ $\operatorname{Hom}_{\mathcal{A}}(a,b) \to \operatorname{Hom}_{\mathcal{B}}(F(a), F(b))$ is a group homo.

② F is called <u>exact</u> if F is additive and F preserves kernel, cokernel.

Rmk. ① An additive functor preserves direct sum; i.e. $F(a \oplus b) \cong F(a) \oplus F(b)$

② An exact functor preserves short exact sequences.

Rmk. So $\tilde{M}$ can be thought to be obtained by gluing $\widetilde{M[1/f]_0}$ on $D_+(f)$ for different f's.

Thm. 1) $f: M \to N$ be a graded R-mod map. $\widetilde{\operatorname{coker} f} \xrightarrow{\sim} \operatorname{coker} \tilde{f}$ (prove at each stalk)
$\ker \tilde{f} \xleftarrow{\sim} \widetilde{\ker f}$, $\operatorname{coker} \tilde{f} \xrightarrow{\sim} \widetilde{\operatorname{coker} f}$

2) If $0 \to M' \to M \to M'' \to 0$ is exact in $\operatorname{Mod}_R^{gr}$

[This includes the hypothesis that all the maps are graded]

Then $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is also exact in $\operatorname{Mod}_{\mathcal{O}_X}$

3) $(\bigoplus_{i \in I} M_i)^{\sim} \xleftarrow{\sim} \bigoplus_{i \in I} \tilde{M}_i$ in $\operatorname{Mod}_{\mathcal{O}_X}$ for any set I.

i.e. $M \mapsto \tilde{M}$ is an exact functor.

Thm. 1) $\tilde{M} = 0 \iff \forall f$ homo of +ve deg $M[1/f]_0 = 0$

$\iff$ Given a covering $X = \bigcup_{i \in I} D_+(f_i)$, $M[1/f_i]_0$ $\forall i$

$\iff$ Given a covering $X = \bigcup_{i \in I} D_+(f_i)$, for any f_i and any homogeneous $m \in M$ s.t. $\deg f_i \mid \deg m$, $\exists\, n = n(m, f_i)$ s.t. $f_i^n \cdot m = 0$

[A special case is when one can choose f_i's of deg 1 s.t. $X = \bigcup_{i=0}^{r} D(f_i)$]

2) For $M \in \operatorname{Mod}_R^{gr}$, $\tilde{i}$ is an isom.

$i: \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \hookrightarrow M$.

Pf $\tilde{M} = 0 \iff \forall f$ homo of +ve deg $\tilde{M}|_{D_+(f)} = 0$

$i: \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \hookrightarrow$ ...

Pf. $\widetilde{M} = 0 \iff \forall f$ homo of +ve deg $\widetilde{M}|_{D_+(f)} = 0$

$\iff$ ($\widetilde{M}$ q.coh) ——— " ——— $M[1/f]_0 = 0$

$\iff$ for a collection $\{f_i\}_{i \in I}$, f_i homo of +ve deg s.t $X = \bigcup D_+(f_i)$, $M[1/f_i]_0 = 0$

Now assume $X = \bigcup_{i=1}^{r} D(f_i)$

$M[1/f_i]_0 = \{ m/f_i^n \mid \deg m = n \deg f_i \}$

$M[1/f_i]_0 = 0 \iff$ for $m \in M_\lambda$ s.t $\deg f_i \mid \lambda$

$m / f_i^{\lambda/\deg f_i} = 0 \quad \in M[1/f_i]_0 \subseteq M[1/f_i]$

$\iff f_i^{n_i} m = 0$ for some n_i.

2) Consider the exact seq in Mod_R^{gr}

$$0 \to \bigoplus_{\lambda \in \mathbb{N}} M_\lambda \to \underbrace{\bigoplus_{\lambda \in \mathbb{Z}} M_\lambda}_{M} \to Q \to 0$$

Every nonzero homo elt in Q lifts to a non-zero homo elt, m, of $-$ve deg in M

Now for every homo elt, f, of R of +ve deg

$\deg f^n . m \geq 0$ for $n >> 0 \Rightarrow f^n . \bar{m} = 0$ ($\bar{m}$ im in Q)

$\Rightarrow \widetilde{Q} = 0$

$$\Rightarrow \quad 0 \to \widetilde{\bigoplus_{\lambda \in \mathbb{N}} M_\lambda} \to \widetilde{M} \to 0 \text{ is exact.}$$

So far we have constructed an exact functor

$$\mathrm{Mod}_R^{gr} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Proj}(R))$$

Under some finiteness assumption on R we will construct an exact functor T_* : $\mathrm{QCoh}(\mathrm{Proj}\, R) \to \mathrm{Mod}_R^{gr}$ s.t

$F \xrightarrow{\sim} \widetilde{\Gamma_* F}$ is a natural isom.

The finiteness assumption on R: From now on, unless otherwise stated we assume R is an $\mathbb{N}$-graded ring which is finitely gen over the degree zero piece R_0.

This is equivalent to saying one of the following.

ⓐ $\exists$ homogeneous elts of +ve degree $g_1, g_2 \ldots g_r \in R$ s.t the R_0-alg map

$$R_0[x_1 \ldots x_r] \to R \text{ is sur.}$$
$$x_i \longmapsto g_i$$

ⓑ The ideal R_+ is generated by finitely many homogeneous elts.
(for a choice $g_1 \ldots g_r$ as above, $R_+ = (g_1 \ldots g_r)$)

You will notice that the following property of R is what we use in the sequel.

Prop. Let R be as before. $\exists$ $d \in \mathbb{N}_{>0}$ and homogeneous elements of R $h_1 \ldots h_r$; each of degree d such that

$$\mathrm{Proj}(R) = \bigcup_{i=1}^{r} D_+(h_i)$$

Pf. Let $g_1, g_2 \ldots g_r$ be as before the alg generators

Pf. Let $g_1, g_2, \dots g_r$ be as before the alg generators

set $h_i = g_i^{\prod_{j \neq i} \deg g_j}$; $d = \deg(g_1)\deg(g_2) \dots \deg(g_r)$

Then $\deg(h_i) = d \ \forall i$. Since $\sqrt{(h_1, \dots, h_r)} = \sqrt{(g_1, \dots, g_r)} = R_+$

$\bigcup_{i=1}^{r} D_+(h_i) = \bigcup_{i=1}^{r} D_+(g_i) = \mathrm{Proj}(R)$.

Locally free sheaves:

Def X be a scheme, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ is called locally free if $\exists$ an open covering $X = \bigcup_{i \in I} U_i$ s.t. $\mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^{\oplus J}$ for some set J; $|J|$ is called the rank of $\mathcal{F}|_{U_i}$.

Def: A locally free sheaf of constant rank 1 is called an invertible sheaf. In literature invertible sheaves are called line bundles.

Prop $\mathcal{L}$ be an invertible sheaf. Denote $\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$

Then the natural map $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \mathcal{O}_X$ is an isom.

$\varphi \otimes f \longrightarrow \varphi(f)$

$\mathcal{L}^\vee$ is also denoted $\mathcal{L}^{-1}$.

Serre twist

R as before. From now on fix choices $d \in \mathbb{N}_{>0}$ and homogeneous elts $g_1, g_2, \dots g_r$ of deg d s.t. $\mathrm{Proj}(R) = \bigcup_{i=1}^{r} D_+(g_i)$

Set $X = \mathrm{Proj}(R)$

Def. • For $M \in \mathrm{Mod}_R^{gr}$, $M(n) \in \mathrm{Mod}_R^{gr}$ is the object whose underlying R-mod is M, but

$$M(n)_m = M_{n+m}$$

• $\mathcal{O}_X(n) := \widetilde{R(n)}$ • For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

$\mathcal{F}(n)$ is called the n-th Serre twist of $\mathcal{F}$.

Note There is a map $M_0 \longrightarrow \Gamma(X, \widetilde{M})$ and so

$M_n = (M(n))_0 \longrightarrow \Gamma(X, \widetilde{M(n)})$

Prop (i) There are natural maps $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m)$ and $\mathcal{F}(n) \otimes \mathcal{O}_X(m) \longrightarrow \mathcal{F}(n+m)$

(ii) For any $n \in \mathbb{Z}$, $\mathcal{O}_X(nd)$ is invertible. The map $\mathcal{O}_{D_+(g_i)} \longrightarrow \mathcal{O}_X(nd)|_{D_+(g_i)}$, $1 \longmapsto g_i^n$ is an isom. $\mathcal{O}_X(nd)^{-1} \xrightarrow{\sim} \mathcal{O}_X(-nd)$

(iii) For any qcoh $\mathcal{F}$, $m, n \in \mathbb{Z}$, $\mathcal{F}(m) \otimes \mathcal{O}_X(nd) \longrightarrow \mathcal{F}(m+nd)$ is an isom.

Pf. i) $\mathcal{O}_X(n)|_{D_+(g)} \xrightarrow{\sim} (R(n)[1/g])_0 \xrightarrow{\sim} (R[1/g])_n$

Thus $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n+m)$ is given by

$$R[1/g]_n \otimes_{R[1/g]_0} R[1/g]_m \longrightarrow R[1/g]_{n+m}$$

Thus $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \to \mathcal{O}_X(n+m)$ ...

$R[1/g]_n \otimes_{R[1/g]_0} R[1/g]_m \longrightarrow R[1/g]_{n+m}$

Rest is clear.

(ii) For g_i as before, $\mathcal{O}_X(nd)|_{D_+(g_i)} \xrightarrow{\sim} \widetilde{R[1/g_i]_{(nd)}}$

Note that $R[1/g_i]_0 \longrightarrow R[1/g_i]_{nd}$ is an isom of $R[1/g]_0$ mod
$1 \longrightarrow g_i^n$

Let's check surjectivity. Any elt in $R[1/g_i]_{nd}$ is of the form α/g_i^t s.t $\deg \alpha - td = nd$

Rewrite $\alpha/g_i^t = \dfrac{\alpha}{g_i^t g_i^n} \cdot g_i^n$.

(iii) Suffices to take $\mathcal{F} = \mathcal{O}_X$. We check that the induced map $R[1/g]_m \otimes_{R[1/g]_0} R[1/g]_{nd} \longrightarrow R[1/g]_{m+nd}$ is an isom

The inverse is given by $\alpha \longmapsto \dfrac{\alpha}{g_i^{nd}} \otimes g_i^{nd}$

Construction of Γ_*:

$R, \{g_i\}$ be as before. $X = \mathrm{Proj}(R)$

Def. Given $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, set $\Gamma_*\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$

Prop (i) $\Gamma_*\mathcal{O}_X$ is a ring. $\Gamma_*\mathcal{F}$ is an $\Gamma_*\mathcal{O}_X$-module

(ii) There is a map of graded rings $R \to \Gamma_*\mathcal{O}_X$.

(iii) Thus $\Gamma_*\mathcal{F}$ is naturally an R-mod.

(iv) There is an $\mathcal{O}_X$-lin map $\widetilde{\Gamma_*\mathcal{F}} \longrightarrow \mathcal{F}$

Pf (i), (ii), (iii) clear

- We describe $\widetilde{\Gamma_*\mathcal{F}}|_{D_+(g_i)} \longrightarrow \mathcal{F}|_{D_+(g_i)}$ for each i

Since $\widetilde{\Gamma_*\mathcal{F}}$ is qcoh, enough to describe

$\Gamma_*\mathcal{F}[1/g_i]_0 \longrightarrow \mathcal{F}(D_+(g_i))$

Any $\alpha \in \Gamma_*\mathcal{F}[1/g_i]_0$ is of the form β/g_i^n where $\beta = \Gamma(X, \mathcal{F}(nd))$ for ... There is a map

Any $\alpha \in \Gamma_* \mathcal{F}[1/g_i]_0$ is of the form β/g_i^n where $\beta = \Gamma(X, \mathcal{F}(nd))$ for some $n \in \mathbb{N}$. $g_i^{-n} \in \Gamma(X, \mathcal{O}_X(-nd))$. There is a map

$$\Gamma(D_+(g_i), \mathcal{F}(nd)) \otimes \Gamma(D_+(g_i), \mathcal{O}_X(-nd)) \xrightarrow{\varphi} \Gamma(D_+(g_i), \mathcal{F})$$

Send α to $\varphi(\beta|_{D_+(g_i)} \otimes g_i^{-n})$

One needs to prove that this map is well defined, i.e. does not depend on the choice of the representation $\beta = \alpha/g_i^n$

- The maps $\widetilde{\Gamma_* \mathcal{F}}|_{D_+(g_i)} \to \mathcal{F}|_{D_+(g_i)}$ glue to produce
- a map $\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$. (check gluing at stalks).

Thm Assume R is a finitely generated R_0 alg. $g_1, g_2, \ldots, g_k$ be deg d elts of R s.t $\mathrm{Proj}(R) = \bigcup_{i=1}^{k} D_+(g_i)$. For any $\mathcal{F} \in \mathrm{Qcoh}(X)$

$\widetilde{\Gamma_* \mathcal{F}} \to \mathcal{F}$ is an isom of $\mathcal{O}_X$-mods.

We need the notion of non-vanishing locus of a section of an invertible sheaf.

Def. $\mathcal{L}$ invertible sheaf on Y. $s \in \Gamma(Y, \mathcal{L})$.

Define $D_s = \{ y \in Y \mid s_y \notin m_y \mathcal{L}_y \}$ m_y is the max ideal of $\mathcal{O}_{Y,y}$.

Rmk: D_s is open

Eg. $X = \mathrm{Proj}(R)$. Take $f = R_{nd}$. Then f can be viewed as $f \in \Gamma(X, \mathcal{O}_X(nd))$. Then $D_f = D_+(f)$

Needed lemma

Lemma: Y quasi compact scheme, $\mathcal{G} \in \mathrm{Qcoh}(Y)$, $\mathcal{L}$ invertible, $s \in \Gamma(Y, \mathcal{L})$. Let

(i) Given $t_1 \in \Gamma(Y, \mathcal{G})$ if $t_1|_{D_s} = 0$,
$\exists\ n \in \mathbb{N}$ s.t $t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(ii) Assume Y is quasi-separated.

[illegible] $\exists\ n \in \mathbb{N}$ $\tilde{f} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

Given $t \in \Gamma(D_s, \mathcal{G})$, $\exists\ n \in \mathbb{N}$, $\tilde{t} \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

s.t $\quad t \otimes s^n = \tilde{t}|_{D_s}$

$\qquad \Gamma(D_s, \mathcal{G} \otimes \mathcal{L}^n)$

Pf: (i) Choose an affine open covering $Y = \bigcup_{j=1}^{m} U_j$ s.t $\mathcal{L}|_{U_j} \cong \mathcal{O}_{U_j}$. Fix isomorphisms $\mathcal{L}|_{U_j} \to \mathcal{O}_{U_j}$, denote the image of $s|_{U_j}$ by f_j

$t_1|_{U_j \cap D_s} = 0$, $U_j \cap D_s = D_{U_j}(f_j)$ [This means the basic affine open given by f_j inside U_j]

Since U_j is affine and $\mathcal{G}|_{U_j}$ is qcoh $f_j^{n_j} \cdot t_1 = 0 \in \Gamma(U_j, \mathcal{G}|_{U_j})$ for some n_j

$\Rightarrow \quad t_1 \otimes s^{n_j}|_{U_j} = 0 \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_j})$

Take $n = \max\{n_1, \dots, n_m\}$, Then $t_1 \otimes s^n|_{U_j} = 0 \ \forall j$

$\Rightarrow t_1 \otimes s^n = 0 \in \Gamma(Y, \mathcal{G} \otimes \mathcal{L}^n)$

(ii) Since $D_s \cap U_j = D(f_j)$ in U_j and $\mathcal{G}(D_s \cap U_j) = \mathcal{G}(U_j)[1/f_j]$ [$\because \mathcal{G}$ is qcoh]

$\exists\ n_j$ s.t $f_j^{n_j} \cdot t$ is the restriction of a section in $\Gamma(\mathcal{G}, U_j)$ to $D(f_j) = D_s \cap U_j$. This means $\exists\ t_j \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_j})$ s.t $t \otimes s^{n_j} = t_j|_{D_s \cap U_j}$

Let $n_0 = \max\{n_1, n_2, \dots, n_r\}$

Set $t'_j = t_j \otimes s^{n_0 - n_j} \in \Gamma(U_j, \mathcal{G} \otimes \mathcal{L}^{n_0})$

On $U_{j_1} \cap U_{j_2} \cap D_s$ $\quad t'_{j_1} = t'_{j_2} = t|_{U_{j_1} \cap U_{j_2} \cap D_s} \otimes s^{n_0}|_{U_{j_1} \cap U_{j_2} \cap D_s}$

Since Y is quasi-separated. $U_{j_1} \cap U_{j_2}$ is quasi-compact.

So by (i) $\exists\ n_{j_1 j_2}$ s.t $\quad (t'_{j_1} - t'_{j_2}) \otimes s^{n_{j_1 j_2}}|_{U_{j_1} \cap U_{j_2}} = 0$

Take $N = \max_{j_1 j_2} \{n_{j_1 j_2}\}$. Thus $t'_{j_\alpha} \otimes s^N$ glue to produce a section $\tilde{t} \in \Gamma(X, \mathcal{G} \otimes \mathcal{L}^{n_0 + N})$

$\tilde{t}|_{D_s} = t \otimes s^{n_0 + N}$ (check the restrictions to each U_j).

Back to (†)

injectivity. If s/g_i^n goes to zero

$s|_{D_+(g_i)} = 0 \Rightarrow \exists\, n_i$ s.t $s \cdot g_i^{n_i} = 0 \in \Gamma(X, \mathcal{F}(n_i d))$

$\Rightarrow s/g_i^n = 0$ in $(\Gamma_* \mathcal{F}[1/g_i])_0$

Surjectivity. Note $X = \bigcup D_+(g_i)$

$D_+(g_\alpha) \cap D_+(g_\beta) = D_+(g_\alpha g_\beta)$

So we can apply (ii) of Lemma above on X.

Given $t \in \mathcal{F}(D_+(g_i))$. $\exists\, n_i \in \mathbb{N}$ s.t

$t \otimes g_i^{n_i} = \tilde{t}|_{D_+(g_i)}$ for some $\tilde{t} \in \Gamma(X, \mathcal{F}(n_i d))$

Then $\tilde{t}/g_i^{n_i} \in (\Gamma_* \mathcal{F}[1/g_i])_0$ with its image in $\mathcal{F}|_{D_+(g_i)}$ being t.

Thm: Assume that $\exists$ homog elts of +ve deg $g_1, g_2 \ldots g_r$ s.t $R = R_0[g_1, g_2 \ldots g_r]$ and $\mathrm{Proj}(R) = X$ is locally noeth. Given $\mathcal{F} \in \mathrm{Coh}(X)$, $\exists$ a finitely gen $M \in \mathrm{Mod}_R^{gr}$ s.t $\tilde{M} \xrightarrow{\sim} \mathcal{F}$

Pf. Choose $n_1, n_2, \ldots n_r$ s.t $\deg g_i^{n_i} = \deg g_j^{n_j} = d \;\forall\, i, j$

Then $X = \bigcup_{i=1}^{r} D_+(g_i^{n_i})$. So $\mathcal{O}_X(d)$ is invertible.

Realize $g_i^{n_i} \in \Gamma(X, \mathcal{O}_X(d))$. Note $D_{g_i^{n_i}} = D_+(g_i^{n_i})$

↑ $g_i^{n_i}$ is thought of in $\Gamma(X, \mathcal{O}_X(d))$

Since $\mathcal{F}$ is coh, $\Gamma(D_+(g_i^{n_i}), \mathcal{F})$ is a f.g $\Gamma(D_+(g_i^{n_i}), \mathcal{O}_X)$ mod. By the lemma above, $\exists\, s_1, s_2, \ldots, s_{r_i} \in \Gamma(X, \mathcal{F}(d_i d))$ such that $\{s_\alpha / g_i^{n_i d_i}\}_{\alpha = 1, \ldots, r_i}$ is a set of gen.

By varying i and possibly increasing d_i, we can find $m \in \mathbb{N}$ & finitely many elts $t_1, t_2, \ldots, t_n \in \Gamma(X, \mathcal{F}(dm))$. s.t $\forall\, i$, $\{t_i / g_i^{n_i m}\}_{i=1,\ldots,n}$ generate the $\Gamma(D_+(g_i^{n_i m}), \mathcal{O}_X)$ mod $\mathcal{F}(D_+(g_i^{n_i}))$.

s.t. $\forall i$, $\{t_i / g_i^{n_i m}\}_{i=1,\dots,n}$ generate the $\Gamma(D_+(g_i^{n_i m}), \mathcal{O}_X)$ mod $\mathcal{F}(D_+(g_i^{n_i}))$.

Let M be the (finitely generated) R-submodule of $\Gamma_* \mathcal{F}$ generated by $t_1, t_2, \dots, t_n$.

Claim: The inclusion map $M \hookrightarrow \Gamma_* \mathcal{F}$ induces an isom $\widetilde{M} \longrightarrow \widetilde{\Gamma_* \mathcal{F}}$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

Pf It's enough to check that for each i the induced map $\Gamma(D_+(g_i^{n_i}), \widetilde{M}) \longrightarrow \Gamma(D_+(g_i^{n_i}), \widetilde{\Gamma_* \mathcal{F}})$ is an isom.

$$\begin{array}{ccc} \Gamma(D_+(g_i^{n_i}), \widetilde{M}) & \longrightarrow & \Gamma(D_+(g_i^{n_i}), \widetilde{\Gamma_* \mathcal{F}}) \\ \uparrow \wr & & \uparrow \wr \\ (M[1/g_i^{n_i}])_0 & \longrightarrow & (\Gamma_* \mathcal{F}[1/g_i^{n_i}])_0 \end{array}$$

The injectivity follows as $M \subseteq \Gamma_* \mathcal{F}$; surjectivity follows from the diag

$$\begin{array}{ccc} (M[1/g_i^{n_i}])_0 & \longrightarrow & \mathcal{F}(D_+(g_i^{n_i m})) \\ & \searrow & \nearrow \\ & & (\Gamma_* \mathcal{F}[1/g_i^{n_i}])_0 \end{array}$$

where the top arrow is surj by construction.

Invertible sheaves:

Def: X be a scheme. A locally free sheaf of rk 1 is called an invertible sheaf.

Prop: i) Let $\mathcal{L}$ be an invertible sheaf. Then

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \mathcal{O}_X \qquad (\varphi \otimes s) \mapsto \varphi(s)$$

is an isom.

ii) For an $\mathcal{O}_X$-mod $\mathcal{F}$, suppose there is an $\mathcal{O}_X$-mod $\mathcal{G}$ and an isom $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{O}_X$. Then $\mathcal{F}$ is invertible.

Pf (i) Given $x \in X$, choose an open nbhd U of x s.t. $\mathcal{L}|_U \xleftarrow{\sim} \mathcal{O}_U$. We have a diag using this isom

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)|_U & \otimes_{\mathcal{O}_U} & \mathcal{L}|_U & \longrightarrow & \mathcal{O}_U \\ \downarrow \wr & & \uparrow \wr & & \| \, id \\ \mathcal{O}_U \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) & \otimes & \mathcal{O}_U & \longrightarrow & \mathcal{O}_U \end{array}$$

Since the bottom row is an isom, we are done.

(ii) Check at stalks.

Prop: X be a scheme. The isom classes of invertible $\mathcal{O}_X$- ... under $\otimes$ operation ...

Prop. X be a scheme. The isom classes of invertible $\mathcal{O}_X$-mods form an abelian group under $\otimes$ operation, denoted $\mathrm{Pic}(X)$ — called the Picard group or the group of invertible sheaf.

Prop\Eg. • R be an $\mathbb{N}$-graded ring. Suppose $\exists\, g_1, g_2 \ldots g_r$ each of deg d such that $\mathrm{Proj}(R) = \bigcup_{i=1}^{r} D_+(g_i)$.
Then for each n, $\mathcal{O}_X(nd)$ is invertible.

- So if $d=1$, each $\mathcal{O}_X(n)$ invertible
- Assume R is gen over R_0 by deg 1 elt as an alg (i.e R standard graded), then $d=1$ and each $\mathcal{O}_X(n)$ is invertible.

Eg: We will see $\mathrm{Pic}(\mathbb{A}^n_{\mathbb{C}}) \simeq \{id\}$, $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow{\sim} \mathbb{Z}\cdot\mathcal{O}(1)$
$\uparrow$
$\mathbb{P}^n_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x_0 \ldots x_n])$

End of 13.11.24 Lecture.

Def. X scheme, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$ is called <u>globally generated</u> if there is a surjection of $\mathcal{O}_X$ mods $\bigoplus_I \mathcal{O}_X \longrightarrow \mathcal{F}$, ($|I|$ need not be finite)

Rmk. Note $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \xleftarrow{\sim} \Gamma(X,\mathcal{G})$. So giving a surj morphism of $\mathcal{O}_X$-mod $\bigoplus_I \mathcal{O}_X \to \mathcal{G}$ is the same as choosing $|I|$ many elts of $\Gamma(X,\mathcal{G})$ such that those generate every stalk $\mathcal{G}_x$, $x \in X$.

Def: An invertible $\mathcal{O}_X$-mod $\mathcal{L}$ is called ample, if for any $\mathcal{F} \in \mathrm{Coh}(X)$ $\exists\, n_{\mathcal{F}} \in \mathbb{N}$ s.t $\forall\, n \geq n_{\mathcal{F}}$,
$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$ is globally generated.

Prop: $\mathcal{L}$ is ample $\Leftrightarrow$ $\mathcal{L}^n$ is ample for some $n \in \mathbb{N}_{>0}$.

Thm: Let $R = \bigoplus_{\lambda \in \mathbb{N}} R_\lambda$ be a noetherian graded ring. $X = \mathrm{Proj}(R)$
If for some $m > 0$, $\mathcal{O}_X(m)$ is invertible, $\mathcal{O}_X(m)$ is ample.

Pf. Recall R is noeth iff R_0 is noeth and $\exists\, g_1, g_2 \ldots g_r$ homo of +ve deg such that $R = R_0[g_1 \ldots g_r]$
Let $d = \mathrm{lcm}(\deg g_1, \deg g_2 \ldots, \deg g_r)$

Lemma. Let d' be a +ve integer multiple of d. Then the graded subring $\bigoplus_{j=0}^{\infty} R_{d'j} \subseteq R$ is generated as an R_0-algebra by elts of degree d'.

Pf. See

By the Prop before it's enough to show that $\mathcal{O}_X(\cdot d m r) \cong \mathcal{O}_X(m)^{\otimes dr}$ is ample – We show this.

Recall $R_{\infty}^{(dmr)}$ is the graded ring whose underlying ring is $\bigoplus_{j=0} R_{dmrj}$ but whose λ-th graded piece is $R_{dmr\lambda}$. By the lemma above $R^{(mdr)}$ is generated over it's degree 0-piece - R_0, by it's set of degree 1 elts. So $R^{(mdr)}$ is a standard graded ring.

The above modifications with the following Prop often allows to reduce a problem about non-standard graded ring to a standard graded ring.

<u>Prop</u>: Let S be an $\mathbb{N}$-graded ring, $a \in \mathbb{N}$, the inclusion map of rings $S^{(a)} \hookrightarrow S$ (it takes deg j elts to deg dj elts) induces an isom $\varphi: \operatorname{Proj} S \to \operatorname{Proj}(S^{(a)})$ such that $\varphi^*(\mathcal{O}_{S^{(a)}}(1)) \cong \mathcal{O}_S(a)$

Pf: Ex.

Returning to the proof of ampleness, take $a = mdr$.

Consider $R^{(a)} \hookrightarrow R$.

<u>Claim</u>: If S is a standard graded noeth ring. $\mathcal{O}_{\operatorname{Proj}(S)}(1)$ is ample.

<u>Pf.</u> Given a coh sheaf $\mathcal{G}$, choose a finitely generated S mod N such that $\mathcal{F} \cong \tilde{N}$. Choose homog elts $n_1, n_2, \ldots, n_s$ of N that generates N. Permuting the order assume $\deg n_1 \leq \deg n_2 \leq \cdots \leq \deg n_s$.

<u>Claim</u> For $\lambda \geq \lambda' \geq \deg n_s$, $N_\lambda = S_{\lambda-\lambda'} \cdot N_{\lambda'}$

<u>Pf.</u> For $\lambda \geq \deg n_s$, choose $x \in N_\lambda$.

$x = f_1 n_1 + \cdots + f_r n_r$, $\deg f_i = \deg x - \deg n_i$

So f_i is sum of pdt of $\deg x - \deg n_i$ monomials in the deg 1 generators of S. So each f_i can be written as $f_i = \sum \tilde{f}_j^i \cdot q_j$, where $\deg \tilde{f}_j^i = \deg n_s - \deg n_i$, $\deg q_j = \deg x - \deg n_s$, Thus $x = \sum_j \sum_i (\tilde{f}_j^i \cdot n_i) \cdot q_j$

$\deg q_j = \deg x - \deg n_i$, Thus $x = \sum_i \sum_j (\tilde{f}^i_j \cdot n_i) \cdot q_j$

$\deg \tilde{f}^i_j \cdot n_i = \deg n_s \quad \forall\, i,j$. So $N_\lambda = S_{\lambda - \deg n_s} \cdot N_{\deg n_s}$

So $N_\lambda = S_{\lambda-\lambda'} \cdot S_{\lambda' - \deg n_s} N_{\deg n_s}$

$= S_{\lambda-\lambda'} \cdot N_{\lambda'}$

Claim: For $\lambda \geq \deg n_s$; $\tilde{N}(\lambda)$ is glob gen.

Pf. Take $\lambda \geq \deg n_s$.

- Since S is noeth, N f.g, $\exists$ homo elts $\alpha_1, \alpha_2, \ldots, \alpha_t$ generating N_λ as an S_0-mod.

Consider the graded map

$$(*) \quad \bigoplus^t S \longrightarrow N(\lambda)$$
$$1 \longmapsto \alpha_i$$

By the claim above the map is surj onto $\bigoplus_{j \geq \lambda} N_j$.

So the cokernel is annihilated by $(S_+)^{\deg s}$.

So the sheaf given by cokernel ~~is~~ on $\operatorname{Proj} S$ is 0.

Thus taking $\sim$ in $*$, get a surj map

$$\bigoplus^t \mathcal{O}_{\operatorname{Proj}(S)} \longrightarrow \widetilde{N(\lambda)} \longrightarrow 0$$

Claim: Since S is standard graded $\widetilde{N(\lambda)} \xleftarrow{\sim} \tilde{N} \otimes \mathcal{O}_{\operatorname{Proj}(S)}(\lambda)$

Pf. Ex.

Recall $a = \dim R$ and $R^{(a)} \hookrightarrow R$ induces an isom

$\varphi : \operatorname{Proj}(R) \longrightarrow \operatorname{Proj}(R^{(a)})$ such that $\varphi^*(\mathcal{O}_{R^{(a)}}(1))$
$= \mathcal{O}_R(a)$

- Since φ is an isom, pull back of an ample sheaf is ample. Thus $\mathcal{O}_R(\dim R)$ is ample $\Rightarrow \mathcal{O}(d)$ is ample.